

# THE VARIATION OF THE MAXIMAL FUNCTION OF A RADIAL FUNCTION

HANNES LUIRO

**Abstract.** It is shown for the non-centered Hardy-Littlewood maximal operator  $M$  that  $\|DMf\|_1 \leq C_n \|Df\|_1$  for all radial functions in  $W^{1,1}(\mathbb{R}^n)$ .

## 1. INTRODUCTION

The non-centered Hardy-Littlewood maximal operator  $M$  is defined by setting for  $f \in L^1_{loc}(\mathbb{R}^n)$  that

$$Mf(x) = \sup_{B(z,r) \ni x} \frac{1}{|B(z,r)|} \int_{B(z,r)} |f(y)| dy =: \sup_{B(z,r) \ni x} \oint_{B(z,r)} |f(y)| dy \quad (1.1)$$

for every  $x \in \mathbb{R}^n$ . The centered version of  $M$ , denoted by  $M_c$ , is defined by taking the supremum over all balls centered at  $x$ . The classical theorem of Hardy, Littlewood and Wiener asserts that  $M$  (and  $M_c$ ) is bounded on  $L^p(\mathbb{R}^n)$  for  $1 < p \leq \infty$ . This result is one of the cornerstones of the harmonic analysis. While the absolute size of a maximal function is usually the principal interest, the applications in Sobolev-spaces and in the potential theory have motivated the active research of the regularity properties of maximal functions. The first observation was made by Kinnunen who verified [Ki] that  $M_c$  is bounded in Sobolev-space  $W^{1,p}(\mathbb{R}^n)$  if  $1 < p \leq \infty$ , and inequality

$$|DM_c f(x)| \leq M_c(|Df|)(x) \quad (1.2)$$

holds for all  $x \in \mathbb{R}^n$ . The proof is relatively simple and inequality (1.2) (and the boundedness) holds also for  $M$  and many other variants.

The most challenging open problem in this field is so called ' $W^{1,1}$ -problem': Does it hold for all  $f \in W^{1,1}(\mathbb{R}^n)$ , that  $Mf \in W^{1,1}(\mathbb{R}^n)$  and

$$\|DMf\|_1 \leq C_n \|Df\|_1 ?$$

This problem has been discussed (and studied) for example in [AlPe], [CaHu], [CaMa], [HO], [HM], [Ku] and [Ta]. The fundamental obstacle is that  $M$  is not bounded in  $L^1$  and therefore inequality (1.2) is not enough to solve the problem. In the case  $n = 1$  the answer is known to be positive, as was

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proved by Tanaka [Ta]. For  $M_c$  the problem turns out to be very complicated also when  $n = 1$ . However, Kurka [Ku] managed to show that the answer is positive also in this case.

The goal of this paper is to develop technology for  $W^{1,1}$ -problem in higher dimensions, where the problem is still completely open. The known proofs in the one-dimensional case are strongly based on the simplicity of the topology: the crucial trick (in the non-centered case) is that  $Mf$  does not have a strict local maximum outside the set  $\{Mf(x) = f(x)\}$ . This fact is a strong tool when  $n = 1$  but is far from sufficient for higher dimensions.

The formula for the derivative of the maximal function (see Lemma 2.2 or [L]) has an important role in the paper. It says that if  $Mf(x) = \fint_B |f|$ ,  $|f(x)| < Mf(x) < \infty$ , and  $Mf$  is differentiable at  $x$ , then

$$DMf(x) = \fint_B Df(y) dy. \quad (1.3)$$

From this formula one can see immediately the validity of the estimate (1.2) for  $M$ . However, since  $B$  is exactly the ball which gives the maximal average (for  $|f|$ ), it is expected that one can derive from (1.3) much more sophisticated estimates than (1.2). In Section 2 (Lemma 2.2), we perform basic analysis related to this issue. The key observation we make is that if  $B$  is as above, then

$$\int_B Df(y) \cdot (y - x) dy = 0. \quad (1.4)$$

In the background of this equality stands a more general principle, concerning other maximal operators as well: if the value of the maximal function is attained to ball (or other permissible object)  $B$ , then the *weighted* integral of  $|Df|$  over  $B$  is zero for a set of weights depending on the maximal operator. We believe that the utilization of this principle is a key for a possible solution of  $W^{1,1}$ -problem.

As the main result of this paper, we employ equality (1.4) to show that in the case of *radial functions* the answer to  $W^{1,1}$ -problem is positive (Theorem 3.11). Even in this case the problem is evidently non-trivial and truly differs from the one-dimensional case. To become convinced about this, consider the important special case where  $f$  is radially decreasing ( $f(x) = g(|x|)$ , where  $g : [0, \infty) \rightarrow \mathbb{R}$  is decreasing). In this case  $Mf$  is radially decreasing as well and  $Mf(0) = f(0)$ . If  $n = 1$ , these facts immediately imply that  $\|DMf\|_1 = \|Df\|_1$ , but if  $n \geq 2$  this is definitely not the case: the additional estimates are necessary. This type of estimate for radially decreasing functions can be derived from (1.3) and (1.4), saying that

$$|DMf(x)| \leq \frac{C_n}{|x|} \fint_{B(0,|x|)} |Df(y)| |y| dy. \quad (1.5)$$

By using this inequality, the positive answer to  $W^{1,1}$ -problem for radially decreasing functions follows straightforwardly by Fubini Theorem (Corollary 3.1).

For general radial functions, inequality (1.5) turns out to hold only if the maximal average is achieved in a ball with radius comparable to  $|x|$ . To overcome this problem, we study the auxiliary maximal function  $M^I$ , defined for  $f \in L^1_{loc}(\mathbb{R}^n)$  by

$$M^I f(x) = \sup_{x \in B(z,r), r \leq |x|/4} \int_{B(z,r)} |f(y)| dy,$$

and prove (Lemma 3.2) that for all radial  $f \in W^{1,1}(\mathbb{R}^n)$  it holds that

$$\|DM^I f\|_1 \leq C_n \|Df\|_1. \quad (1.6)$$

The proof of this auxiliary result resembles the proof of  $W^{1,1}$ -problem (for  $M$ ) in the case  $n = 1$ . As the first step, we prove by straightforward calculation that for the 'endpoint operator' of  $M^I$ , defined by

$$f_{/4}(x) := \sup_{x \in B(z, |x|/4)} \int_{B(z, |x|/4)} |f(y)| dy, \quad (1.7)$$

it holds that  $\|Df_{/4}\|_1 \leq C \|Df\|_1$  for all  $f \in W^{1,1}(\mathbb{R}^n)$ . Recall again the fact that  $Mf$  does not have a local maximum in  $\{Mf(x) > |f(x)|\}$ , leading to the estimate  $\|DMf\|_1 \leq \|Df\|_1$  in the case  $n = 1$ . As a multidimensional counterpart for radial functions, we show that  $M^I f$  does not have a local maximum in  $\{M^I f(x) > \max\{|f(x)|, f_{/4}(x)\}\}$  and for every  $k \in \mathbb{Z}$  it holds that

$$\int_{\{2^k \leq |y| \leq 2^{k+1}\}} DM^I f(y) dy \leq C_n \int_{\{2^{k-1} \leq |y| \leq 2^{k+2}\}} |Df|(y) dy.$$

Estimate (1.6) can be easily derived from this fact. The main result follows by combining (1.6) and exploiting the estimate (1.5) in  $\{Mf(x) > M^I f(x)\}$ .

**Question.** The analysis presented in this paper raises the interest towards the study of the integrability properties of some *conditional* maximal operators. As an example, (1.3) and (1.4) yield that  $|DMf(x)| \leq \widetilde{M}(|Df|)(x)$ , where  $\widetilde{M}$  is defined for all locally integrable gradient fields  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$\widetilde{M}F(x) = \sup \left\{ \left| \int_{B(z,r)} F \right| : x \in B(z,r), \int_{B(z,r)} F(y) \cdot (y-x) dy = 0 \right\}.$$

It is clear that  $\widetilde{M}F$  is bounded by  $M(|F|)$ , but does it hold that  $\widetilde{M}$  has even better integrability properties than  $M$ ? What about the boundedness in the Hardy-space  $H^1$  or even in  $L^1$ ? Notice that the boundedness of  $\widetilde{M}$  in  $L^1$  would imply the solution to  $W^{1,1}$ -problem. This problem is almost completely open, even in the case  $n = 1$ . Counterexamples would be highly interesting as well.

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## 2. PRELIMINARIES AND GENERAL RESULTS

Let us introduce some notation. The boundary of the  $n$ -dimensional unit ball is denoted by  $S^{n-1}$ . The  $s$ -dimensional Hausdorff measure is denoted by  $\mathcal{H}^s$ . The volume of the  $n$ -dimensional unit ball is denoted by  $\omega_n$  and the  $\mathcal{H}^{n-1}$ -measure of  $S^{n-1}$  by  $\sigma_n$ . The weak derivative of  $f$  (if exists) is denoted by  $Df$ . If  $v \in S^{n-1}$ , then

$$D_v f(x) := \lim_{h \rightarrow 0} \frac{1}{h} (f(x + hv) - f(x)),$$

in the case the limit exists.

**Definition 2.1.** For  $f \in L^1_{loc}(\mathbb{R}^n)$  let

$$\mathcal{B}_x := \{B(z, r) : x \in \bar{B}(z, r), r > 0, \oint_B |f| = Mf(x)\}.$$

It is easy to see that if  $f \in L^1(\mathbb{R}^n)$  and  $|f(x)| < Mf(x) < \infty$ , then  $\mathcal{B}_x \neq \emptyset$ .

The following lemma is the main result of this section. We point out that below (6) is especially useful in the case of radial functions.

**Lemma 2.2.** Suppose that  $f \in W^{1,1}(\mathbb{R}^n)$ ,  $Mf(x) > f(x)$  and  $Mf$  is differentiable at  $x$ . Then

(1) For all  $v \in S^{n-1}$  and  $B \in \mathcal{B}_x$ , it holds that

$$DMf(x) = \oint_B D|f|(y) dy \quad \text{and} \quad D_v Mf(x) = \oint_B D_v |f|(y) dy.$$

(2) If  $x \in B$  for some  $B \in \mathcal{B}_x$ , then  $DMf(x) = 0$ .

(3) If  $x \in \partial B$ ,  $B = B(z, r) \in \mathcal{B}_x$  and  $DMf(x) \neq 0$ , then

$$\frac{DMf(x)}{|DMf(x)|} = \frac{z - x}{|z - x|}.$$

(4) If  $B \in \mathcal{B}_x$ , then

$$\int_B D|f|(y) \cdot (y - x) dy = 0. \quad (2.8)$$

(5) If  $x \in \partial B$ ,  $B = B(z, r) \in \mathcal{B}_x$ , then

$$|DMf(x)| = \frac{1}{r} \oint_B D|f|(y) \cdot (z - y) dy.$$

(6) If  $B \in \mathcal{B}_x$ , then

$$DMf(x) \cdot \frac{x}{|x|} = \frac{1}{|x|} \oint_B D|f|(y) \cdot y dy. \quad (2.9)$$

The proof of Lemma 2.2 is essentially based on the following auxiliary propositions.

**Proposition 2.3.** *Suppose that  $f \in W^{1,1}(\mathbb{R}^n)$ ,  $B$  is a ball,  $h_i \in \mathbb{R}$  such that  $h_i \rightarrow 0$  as  $i \rightarrow \infty$ , and  $B_i = L_i(B)$ , where  $L_i$  are affine mappings and*

$$\lim_{i \rightarrow \infty} \frac{L_i(y) - y}{h_i} = g(y).$$

Then

$$\lim_{i \rightarrow \infty} \frac{1}{h_i} \left( \int_{B_i} f(y) dy - \int_B f(y) dy \right) = \int_B Df(y) \cdot g(y) dy. \quad (2.10)$$

*Proof.* The proof is a simple calculation:

$$\begin{aligned} & \frac{1}{h_i} \left( \int_{B_i} f(y) dy - \int_B f(y) dy \right) = \frac{1}{h_i} \left( \int_{L_i(B)} f(y) dy - \int_B f(y) dy \right) \\ &= \frac{1}{h_i} \left( \int_B f(L_i(y)) - f(y) dy \right) = \int_B \frac{f(y + (L_i(y) - y)) - f(y)}{h_i} dy \\ &\approx \int_B \frac{Df(y) \cdot (L_i(y) - y)}{h_i} dy \rightarrow \int_B Df(y) \cdot g(y) dy, \end{aligned}$$

if  $i \rightarrow \infty$ . □

**Lemma 2.4.** *Let  $f \in W^{1,1}(\mathbb{R}^n)$ ,  $x \in \mathbb{R}^n$ ,  $B \in \mathcal{B}_x$ ,  $\delta > 0$ , and let  $L_h$ ,  $h \in [-\delta, \delta]$ , be affine mappings such that  $x \in L_h(\bar{B})$  and*

$$\lim_{h \rightarrow 0} \frac{L_h(y) - y}{h} = g(y). \quad (2.11)$$

Then

$$\int_B D|f|(y) \cdot g(y) dy = 0. \quad (2.12)$$

*Proof.* Let us denote  $B_h := L_h(B)$ . By Proposition 2.3 it holds that

$$\int_B D|f|(y) \cdot g(y) dy = \lim_{h \rightarrow 0} \frac{1}{h} \left( \int_{B_h} |f|(y) - \int_B |f|(y) \right).$$

Since  $B \in \mathcal{B}_x$  and  $x \in \bar{B}_h$ , the sign of the quantity inside the large parentheses is non-positive for all  $h \in [-\delta, \delta]$ . However, the sign of  $1/h$  depends on the sign of  $h$ . The conclusion is that the above equality is possible only if (2.12) is valid. □

### Proof of Lemma 2.2.

- (1) The claim is counterpart for the formula for  $DM_c f$ , which was first time proved in [L]. Suppose that  $B = B(z, r) \in \mathcal{B}_x$  and let  $B_h :=$

$B(z + hv, r)$ . Then it holds that

$$\begin{aligned} D_v Mf(x) &= \lim_{h \rightarrow 0} \frac{1}{h} (Mf(x + hv) - Mf(x)) \\ &\geq \lim_{h \rightarrow 0} \frac{1}{h} \left( \int_{B_h} |f(y)| dy - \int_B |f(y)| dy \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \int_B |f(y + hv)| - |f(y)| dy \right) = \int_{B_h} D_v |f|(y) dy. \end{aligned}$$

On the other hand, if  $B_h := B(z - hv, r)$ , then

$$\begin{aligned} D_v Mf(x) &= \lim_{h \rightarrow 0} \frac{1}{h} (Mf(x) - Mf(x - hv)) \\ &\leq \lim_{h \rightarrow 0} \frac{1}{h} \left( \int_B |f(y)| dy - \int_{B_h} |f(y)| dy \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \int_B |f(y)| - |f(y + hv)| dy \right) = \int_{B_h} D_v |f|(y) dy. \end{aligned}$$

These inequalities imply the claim.

- (2) If  $B \in \mathcal{B}_x$  and  $x \in B$ , then  $y \in B$  if  $|y - x|$  is small enough, and thus  $Mf(y) \geq Mf(x)$ .
- (3) Let  $B = B(z, r) \in \mathcal{B}_x$ ,  $v \in S^{n-1}$  such that  $v \cdot (z - x) = 0$ , and  $h_i \in (0, \infty)$ ,  $h_i \rightarrow 0$  as  $i \rightarrow \infty$ . Moreover, let us denote  $B_i := B(z, |z - (x + h_i v)|)$ . Then it clearly holds that  $x + h_i v \in \bar{B}_i$  and it is also easy to see that  $B_i = L_i(B)$  for an affine mapping  $L_i$  given by

$$L_i(y) = y + \left( \frac{|z - (x + h_i v)| - |z - x|}{|z - x|} \right) (y - z).$$

By the assumption  $v \cdot (z - x) = 0$  it follows that

$$\lim_{i \rightarrow \infty} \frac{L_i(y) - y}{h_i} = (y - z) \lim_{i \rightarrow \infty} \left( \frac{|z - (x + h_i v)| - |z - x|}{|z - x|} \right) = 0.$$

Therefore, Proposition 2.3 implies that

$$\lim_{i \rightarrow \infty} \frac{1}{h_i} \left( \int_{B_i} |f|(y) dy - \int_B |f|(y) dy \right) = 0.$$

This shows that  $D_v Mf(x) = 0$  for all  $v$  orthogonal to  $(z - x)$ . In particular, it follows that  $DMf(x)$  is parallel to  $z - x$  or  $x - z$ . The final claim follows easily by the fact that  $Mf(x + h(z - x)) \geq Mf(x)$  if  $0 < h \leq 2$ .

- (4) Let  $B \in \mathcal{B}_x$  and  $L_h(y) := y + h(y - x)$ ,  $h \in \mathbb{R}$ . Then it holds that  $L_h$  is affine mapping,  $L_h(x) = x$ , and so  $x \in L_h(B) =: B_h$ , and  $(L_h(y) - y)/h = y - x$  for all  $h \in \mathbb{R}$ . Therefore, Lemma 2.4 implies that

$$\int_B D|f|(y) \cdot (y - x) dy = 0.$$

(5) By combining (1), (3) and (4) the claim follows by

$$\begin{aligned} |DMf(x)| &= DMf(x) \cdot \left( \frac{z-x}{|z-x|} \right) = \int_B D|f|(y) \cdot \left( \frac{z-x}{|z-x|} \right) dy \\ &= \int_B D|f|(y) \cdot \left( \frac{z-y}{|z-x|} \right) dy. \end{aligned}$$

(6) The claim follows from (1) and (4).

□

### 3. $W^{1,1}$ -PROBLEM FOR RADIAL FUNCTIONS

**Radial functions and notation.** In what follows, we will interpret a radial function on  $\mathbb{R}^n$  as a function on  $(0, \infty)$  in a natural way. To be more precise, if  $f \in W_{loc}^{1,1}(\mathbb{R}^n)$  is radial, it is well known fact that there exists continuous function  $\tilde{f} : (0, \infty) \rightarrow \mathbb{R}$  such that  $\tilde{f}$  is weakly differentiable,

$$\int_0^\infty |\tilde{f}'(t)| t^{n-1} dt < \infty,$$

and (by a possible redefinition of  $f$  in a set of measure zero) for all  $t \in (0, \infty)$  it holds that  $f(x) = \tilde{f}(t)$  and  $D_{x/|x|}f(x) = \tilde{f}'(t)$  if  $|x| = t$ . In what follows, we will simplify the notation and use  $f$  to denote  $\tilde{f}$  as well. To avoid the possibility of misunderstanding, we usually use variable  $t$  and notation  $f'$  (instead of  $Df$ ) when we are actually working with  $\tilde{f}$ . We also say that  $f$  is radially decreasing if  $f$  is radial and  $f(t_1) \leq f(t_2)$  if  $t_1 > t_2$ . Notice also that if  $f$  is radial then  $Mf$  is also radial.

The following result is an easy consequence of Lemma 2.2.

**Corollary 3.1.** *If  $f \in W^{1,1}(\mathbb{R}^n)$  is radially decreasing, then  $DMf \in W^{1,1}(\mathbb{R}^n)$  and  $\|DMf\|_1 \leq C_n \|Df\|_1$ .*

*Proof.* Since  $f$  is radially decreasing, it is easy to show (the rigorous proof is left to the reader) that if  $Mf(x) \neq 0$  and  $B \in \mathcal{B}_x$ , then  $0 \in \bar{B}$  and  $\bar{B} \subset \bar{B}(0, |x|)$ . Especially, we get by Lemma 2.2, (6), that

$$|DMf(x)| \leq \frac{C_n}{|x|} \int_{B(0, |x|)} |Df(y)| |y| dy. \quad (3.13)$$

Then the claim follows by Fubini theorem:

$$\begin{aligned}
& \int_{\mathbb{R}^n} \left( \frac{1}{|x|} \int_{B(0,|x|)} |Df(y)| |y| dy \right) dx \\
&= \int_{\mathbb{R}^n} |Df(y)| |y| \left( \int_{\mathbb{R}^n} \frac{\chi_{B(0,|x|)}(y)}{\omega_n |x|^{n+1}} dx \right) dy \\
&= \int_{\mathbb{R}^n} |Df(y)| |y| \left( \int_{\{x: |x| \geq |y|\}} \frac{1}{\omega_n |x|^{n+1}} dx \right) dy \\
&= \int_{\mathbb{R}^n} |Df(y)| |y| \left( \int_{S^{n-1}} \int_{|y|}^{\infty} \frac{1}{\omega_n t^{n+1}} t^{n-1} dt d\mathcal{H}^{n-1} \right) dy \\
&= \frac{\sigma_n}{\omega_n} \int_{\mathbb{R}^n} |Df(y)| |y| \left( \int_{|y|}^{\infty} \frac{1}{t^2} dt \right) dy \\
&= \frac{\sigma_n}{\omega_n} \int_{\mathbb{R}^n} |Df(y)| dy.
\end{aligned}$$

□

In the case of general radial functions, (1.5) is in general valid (and useful) only for those  $x$  for which the radius of  $B \in \mathcal{B}_x$  is comparable to  $|x|$ . As it was explained in the introduction, the main auxiliary tool in the case of general radial functions is the following result (recall the definition of  $M^I$  in the introduction):

**Lemma 3.2.** *If  $f \in W^{1,1}(\mathbb{R}^n)$  is radial, then  $M^I f \in W^{1,1}(\mathbb{R}^n)$  and  $\|DM^I f\|_1 \leq C_n \|Df\|_1$ .*

Before the actual proof of this result, we prove several auxiliary results. The first of them is well known.

**Proposition 3.3.** *Suppose that  $E \subset \mathbb{R}$  is open. Then there exist disjoint intervals  $(a_i, b_i)$  such that  $E = \cup_{i=1}^{\infty} (a_i, b_i)$  and  $a_i, b_i \in \partial E \cup \{-\infty, \infty\}$  for all  $i \in \mathbb{N}$ .*

The following auxiliary result is repeatedly utilized in the proof. The result is well known but we express the proof for readers convenience.

**Lemma 3.4.** *Suppose that  $\Omega \subset \mathbb{R}^n$ ,  $f \in W^{1,1}(\Omega)$  is continuous,  $g : \Omega \rightarrow \mathbb{R}$  is continuous and weakly differentiable in  $E := \{x \in \Omega : g(x) > f(x)\}$ , and  $\int_E |Dg| < \infty$ . Then  $\max\{f, g\}$  is weakly differentiable in  $\Omega$  and*

$$D(\max\{f, g\}) = \chi_E Dg + \chi_{\Omega \cap E^c} Df.$$

*Proof.* Suppose that  $\phi$  is a smooth test function, compactly supported in  $\Omega$ ,  $1 \leq i \leq n$ ,  $L(t) = p + te_i$ ,  $p \in \mathbb{R}^n$ , and let  $L$  denote the line  $L(\mathbb{R})$ . By Proposition 3.3,  $E \cap L$  can be written as a union of disjoint and open (in  $\Omega \cap L$ ) line segments  $E_j = L((a_j, b_j))$ ,  $j \in \mathbb{N}$ , such that  $L(a_j), L(b_j) \in \partial E$  (with



respect to  $\Omega \cap L$ ) or  $a_j = -\infty$  or  $b_j = \infty$ . In particular,  $f(L(a_j)) = g(L(a_j))$  if  $a_j \neq -\infty$  and  $f(L(b_j)) = g(L(b_j))$  if  $b_j \neq \infty$ . Since  $\phi$  is compactly supported, it follows that

$$\begin{aligned} f(L(a_j))\phi(L(a_j)) &= g(L(a_j))\phi(L(a_j)) \text{ and} \\ f(L(b_j))\phi(L(b_j)) &= g(L(b_j))\phi(L(b_j)) \text{ for all } j \in \mathbb{N}. \end{aligned}$$

Therefore, by using the assumptions for  $g$ , it holds that

$$\begin{aligned} \int_{E_j} g(D_i \phi) d\mathcal{H}^1 &= \int_{E_j} D_i(g\phi) d\mathcal{H}^1 - \int_{E_j} (D_i g)\phi d\mathcal{H}^1 \\ &= g(L(b_j))\phi(L(b_j)) - g(L(a_j))\phi(L(a_j)) - \int_{E_j} (D_i g)\phi d\mathcal{H}^1 \\ &= f(L(b_j))\phi(L(b_j)) - f(L(a_j))\phi(L(a_j)) - \int_{E_j} (D_i g)\phi d\mathcal{H}^1 \\ &= \int_{E_j} D_i(f\phi) d\mathcal{H}^1 - \int_{E_j} (D_i g)\phi d\mathcal{H}^1 \\ &= \int_{E_j} (D_i f)\phi + f(D_i \phi) - (D_i g)\phi d\mathcal{H}^1 \end{aligned}$$

for all  $j \in \mathbb{N}$ . Then

$$\begin{aligned} \int_{\Omega \cap L} \max\{f, g\}(D_i \phi) d\mathcal{H}^1 &= \int_{E \cap L} g(D_i \phi) d\mathcal{H}^1 + \int_{\Omega \cap E^c \cap L} f(D_i \phi) d\mathcal{H}^1 \\ &= \sum_{j=1}^{\infty} \int_{E_j} g(D_i \phi) d\mathcal{H}^1 + \int_{\Omega \cap E^c \cap L} f(D_i \phi) d\mathcal{H}^1 \\ &= \int_{E \cap L} (D_i f)\phi + f(D_i \phi) - (D_i g)\phi d\mathcal{H}^1 + \int_{\Omega \cap E^c \cap L} f(D_i \phi) d\mathcal{H}^1 \\ &= \int_{\Omega \cap L} f(D_i \phi) d\mathcal{H}^1 + \int_{E \cap L} (D_i f)\phi d\mathcal{H}^1 - \int_{E \cap L} (D_i g)\phi d\mathcal{H}^1 \\ &= - \int_{\Omega \cap L} (D_i f)\phi d\mathcal{H}^1 + \int_{E \cap L} (D_i f)\phi d\mathcal{H}^1 - \int_{E \cap L} (D_i g)\phi d\mathcal{H}^1 \\ &= - \int_{\Omega \cap E^c \cap L} (D_i f)\phi d\mathcal{H}^1 - \int_{E \cap L} (D_i g)\phi d\mathcal{H}^1 \\ &= - \int_{\Omega \cap L} (\chi_E D_i g + \chi_{\Omega \cap E^c} D_i f)\phi d\mathcal{H}^1. \end{aligned}$$

This implies the claim.  $\square$

**Definition 3.5.** Let  $f : \Omega \rightarrow \mathbb{R}$ , where  $\Omega \subset \mathbb{R}$  is open. We say that  $x$  is a local strict maximum of  $f$  in  $(a, b) \subset \Omega$ ,  $-\infty \leq a < b \leq \infty$ , if there exist  $a', b' \in (a, b)$  such that  $a' < x < b'$ ,  $f(t) \leq f(x)$  if  $t \in (a', b')$ , and  $\max\{f(a'), f(b')\} < f(x)$ .

**Proposition 3.6.** *Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and  $c \in (a, b)$  such that  $f(c) > \max\{f(a), f(b)\}$ . Then  $f$  has a local strict maximum on  $(a, c)$ .*

*Proof.* It is easy to see that now any maximum point  $c$  ( $f(c) = \max f$ ), which is known to exist, is also a local strict maximum of  $f$ .  $\square$

**Proposition 3.7.** *Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and does not have a local strict maximum on  $(a, b)$ . Then there exists  $c \in [a, b]$  such that  $f$  is non-increasing on  $[a, c]$  and non-decreasing on  $[c, b]$ .*

*Proof.* Since  $f$  is continuous, we can choose  $c \in [a, b]$  such that  $f(c) = \min f$ . To show that  $f$  is non-decreasing on  $[c, b]$ , let  $c < y_1 < y_2 < b$  and assume, on the contrary, that  $f(y_2) < f(y_1)$ . This implies that  $f(y_1) > \max\{f(c), f(y_2)\}$ , and thus  $f$  has a strict local maximum on  $(c, y_2)$  by Proposition 3.6. This is the desired contradiction. To show that  $f$  is non-increasing on  $[a, c]$ , let  $a < y_1 < y_2 < c$  and assume, on the contrary, that  $f(y_1) < f(y_2)$ . This implies that  $f(y_2) > \max\{f(y_1), f(c)\}$ , and thus  $f$  has a strict local maximum on  $(y_1, c)$  by Proposition 3.6. This is the desired contradiction.  $\square$

Let us define for  $0 < a \leq b < \infty$  the annular domains

$$A_n(a, b) := A(a, b) := \{x \in \mathbb{R}^n : a < |x| < b\} \quad \text{and} \\ A_n[a, b] := A[a, b] := \{x \in \mathbb{R}^n : a \leq |x| \leq b\}.$$

**Lemma 3.8.** *If  $f \in W^{1,1}(\mathbb{R}^n)$  is radial, then  $Mf$  does not have a local strict maximum in  $\{t \in (0, \infty) : Mf(t) > f(t)\}$ .*

*Proof.* Suppose, on the contrary, that  $t_0 \in (0, \infty)$  is a local strict maximum of  $Mf$  and  $Mf(t_0) > f(t_0)$ . Let us choose

$$t^- := \sup\{t < t_0 : Mf(t) < Mf(t_0)\} \quad \text{and} \\ t^+ := \inf\{t > t_0 : Mf(t) < Mf(t_0)\}.$$

By the definition of the local strict maximum, it follows that  $t_0 \in [t^-, t^+]$  and

$$Mf(t) = Mf(t_0) \quad \text{for all } t \in [t^-, t^+]. \quad (3.14)$$

Suppose that  $|x| = t_0$ . Since  $Mf(t_0) > f(t_0)$ , it follows that there exist a ball  $B$  such that  $Mf(t_0) = \mathbf{f}_B |f|$ ,  $x \in \bar{B}$ . Suppose first that  $B \not\subset A[t^-, t^+]$ . In this case there exists  $\varepsilon > 0$  such that  $[t^- - \varepsilon, t^-] \subset \{|y| : y \in \bar{B}\}$  or  $[t^+, t^+ + \varepsilon] \subset \{|y| : y \in \bar{B}\}$ . Especially, it follows by the definition of  $M$  that  $Mf(t) \geq \mathbf{f}_B |f| = Mf(t_0)$  if  $t \in [t^- - \varepsilon, t^-]$  or  $t \in [t^+, t^+ + \varepsilon]$ , respectively. Obviously this contradicts with the choice of  $t^-$  and  $t^+$ . This verifies that  $B \subset A[t^-, t^+]$ . Therefore, it holds by (3.14) that

$$Mf(y) = Mf(t_0) \quad \text{for all } y \in B. \quad (3.15)$$

However,  $f(t_0) < Mf(t_0)$  also implies that there exists a ball  $B'$  with positive radius such that  $B' \subset B$  and  $f < Mf(t_0)$  in  $B'$ . Combining this with (3.15) yields the desired contradiction by

$$\begin{aligned} Mf(t_0) &= \int_B |f| \leq \frac{1}{|B|} \left( \int_{B \setminus B'} |f| + \int_{B'} |f| \right) \\ &< \frac{1}{|B|} \left( \int_{B \setminus B'} Mf + \int_{B'} Mf(t_0) \right) = Mf(t_0). \end{aligned}$$

□

Recall the definition of  $f_{/4}$  (the endpoint operator of  $M^I$ , (1.7)) from the introduction. Before showing the boundedness for  $M^I$ , we have to prove the boundedness for  $f_{/4}$ .

**Proposition 3.9.** *If  $f \in W^{1,1}(\mathbb{R}^n)$ , then  $f_{/4} \in W^{1,1}(\mathbb{R}^n)$  and  $\|Df_{/4}\|_1 \leq C_n \|Df\|_1$ .*

*Proof.* It is easy to check that  $f_{/4}$  is Lipschitz outside the origin. Therefore, it suffices to verify the desired norm estimates for  $Df_{/4}$ . We will exploit Proposition 2.3. If  $x \neq 0$ , we are going to show that if  $h > 0$  is small enough and  $v \in S^{n-1}$ , then

$$\frac{1}{h} |f_{/4}(x) - f_{/4}(x + hv)| \leq C_n \int_{B(x, \frac{|x|}{2})} |Df|(y) dy. \quad (3.16)$$

To show this, we may assume that  $f_{/4}(x) > f_{/4}(x + hv)$ . Suppose that

$$\begin{aligned} f_{/4}(x) &= \int_{B(z, |x|/4)} |f(y)| dy, \quad x \in \bar{B}(z, |x|/4) =: B, \\ g_h(y) &:= x + hv + \frac{|x + hv|}{|x|} (y - x) \quad \text{and} \\ B_h &:= g_h(B) = B(x + hv + \frac{|x + hv|}{|x|} (z - x), |x + hv|/4). \end{aligned}$$

Especially,  $x + hv \in \bar{B}_h$ . Moreover, it is easy to compute that

$$\lim_{h \rightarrow 0} \frac{g_h(y) - y}{h} = \lim_{h \rightarrow 0} \frac{hv + \left( \frac{|x + hv|}{|x|} - 1 \right) (y - x)}{h} = v + \frac{v \cdot x}{|x|^2} (y - x).$$

Then it follows by Proposition 2.3 that

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f_{/4}(x) - f_{/4}(x + hv)}{h} &\leq \lim_{h \rightarrow 0} \frac{1}{h} \left( \int_B |f(y)| dy - \int_{B_h} |f(y)| dy \right) \\ &= \int_B D|f|(y) \cdot \left( v + \frac{v \cdot x}{|x|^2} (y - x) \right) dy \leq \int_B |Df|(y) \left( 1 + \frac{|y - x|}{|x|} \right) dy \\ &\leq \int_B \left( 1 + \frac{1}{4} \right) |Df|(y) dy \leq C_n \int_{B(x, \frac{|x|}{2})} |Df|(y) dy. \end{aligned}$$

This proves (3.16). Then the claim follows (e.g) by using Fubini Theorem: Let us denote below  $B_x = B(x, \frac{|x|}{2})$ . By the above estimate,

$$\begin{aligned} \int_{\mathbb{R}^n} |Df_{/4}(x)| dx &\leq C_n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\chi_{B_x}(y)}{|B_x|} |Df(y)| dx dy \\ &\leq C_n \int_{\mathbb{R}^n} |Df(y)| \left( \int_{\{x: \frac{2|y|}{3} \leq |x| \leq 2|y|\}} |B_x|^{-1} dx \right) dy \leq C'_n \|Df\|_1. \end{aligned}$$

□

The following estimate is well known.

**Proposition 3.10.** *If  $f \in W^{1,1}(\mathbb{R}^n)$  is radial and  $0 < a < b < \infty$ , then*

$$\sigma_n a^{n-1} \int_a^b |f'(t)| dt \leq \int_{A(a,b)} |Df(y)| dy \leq \sigma_n b^{n-1} \int_a^b |f'(t)| dt.$$

**The proof of Lemma 3.2.** Let

$$g(x) = \max\{f_{/4}(x), |f(x)|\}.$$

By Lemma 3.4 and Proposition 3.9 it follows that  $g \in W^{1,1}(\mathbb{R}^n)$  and  $\|Dg\|_1 \leq C_n \|Df\|_1$ . Let

$$E := \{x \in \mathbb{R}^n : M^I f(x) > g(x)\} \text{ and } E_k := E \cap A[2^{-k}, 2^{-k+1}], \quad k \in \mathbb{N}.$$

It is well known that mapping  $M^I f$  is locally Lipschitz in  $E$  and, especially,  $D(M^I f)$  exists in  $E$ . By Lemma 3.4, it suffices to show that  $\int_E |DM^I f| \leq C_n \|Dg\|_1$ .

First observe that since  $|f|$  is radial, it follows that  $M^I f$  and  $g$  are radial as well, and continuous in  $\mathbb{R}^n \setminus \{0\}$ . In particular, if

$$E_k^{\mathbb{R}} := \{|x| : x \in E_k\},$$

then  $x \in E_k$  if and only if  $|x| \in E_k^{\mathbb{R}}$ . Since  $E_k^{\mathbb{R}}$  is open, we can write

$$E_k^{\mathbb{R}} = \cup_{i=1}^{\infty} (a_i, b_i),$$

such that  $a_i < b_i$ ,  $(a_i, b_i)$  are pairwise disjoint and  $a_i, b_i \in \partial E_k^{\mathbb{R}}$ . In the other words,

$$E_k = \bigcup_{i=1}^{\infty} A(a_i, b_i),$$

and (by the definition of  $E_k$ ) for all  $i \in \mathbb{N}$  it holds that

$$M^I f(x) = g(x) \text{ if } |x| = a_i > 2^{-k} \text{ and } M^I f(x) = g(x) \text{ if } |x| = b_i < 2^{-k+1}. \quad (3.17)$$

Moreover, since  $M^I f > f$  in  $E_k$ , Lemma 3.8 says that  $M^I f$  does not have a strict local maximum in  $E_k^{\mathbb{R}}$ . In particular, by Proposition 3.7 there exist  $c_i \in (a_i, b_i)$  such that

$$\begin{aligned} \int_{A(a_i, b_i)} DM^I f(y) dy &\leq \sigma_n b_i^{n-1} \int_{a_i}^{b_i} |(M^I f)'(t)| dt \\ &= \sigma_n b_i^{n-1} (M^I f(a_i) - M^I f(c_i) + M^I f(b_i) - M^I f(c_i)) \\ &\leq \sigma_n b_i^{n-1} (M^I f(a_i) - g(c_i) + M^I f(b_i) - g(c_i)). \end{aligned}$$

Combining this with (3.17) implies that if  $2^{-k} < a_i < b_i < 2^{-k+1}$ , then

$$\begin{aligned} \int_{A(a_i, b_i)} DM^I f(y) dy &\leq \sigma_n b_i^{n-1} (g(a_i) - g(c_i) + g(b_i) - g(c_i)) \\ &\leq \sigma_n b_i^{n-1} \int_{a_i}^{b_i} |g'(t)| dt \leq \left(\frac{b_i}{a_i}\right)^{n-1} \int_{A(a_i, b_i)} |Dg(y)| dy \\ &\leq 2^{n-1} \int_{A(a_i, b_i)} |Dg(y)| dy. \end{aligned}$$

For the case  $a_i = 2^{-k}$  or  $b_i = 2^{-k+1}$ , we employ the fact

$$M^I f(2^{-k}), M^I f(2^{-k+1}) \leq \sup_{y \in A(2^{-k-1}, 2^{-k+2})} g(y)$$

to obtain the estimates ( $a_i = 2^{-k}$  or  $b_i = 2^{-k+1}$ )

$$\begin{aligned} \int_{A(a_i, b_i)} DM^I f(y) dy &\leq \sigma_n b_i^{n-1} (M^I f(a_i) - g(c_i) + M^I f(b_i) - g(c_i)) \\ &\leq \sigma_n b_i^{n-1} \int_{2^{-k-1}}^{2^{-k+2}} |g'(t)| dt \\ &\leq 2^{3(n-1)} \int_{A(2^{-k-1}, 2^{-k+2})} |Dg(y)| dy. \end{aligned}$$

Combining these estimates implies that

$$\begin{aligned} \int_{E_k} |DM^I f(y)| dy &= \sum_{i=1}^{\infty} \int_{A(a_i, b_i)} |DM^I f(y)| dy \\ &\leq 2^{n-1} \sum_{i=1}^{\infty} \left[ \int_{A(a_i, b_i)} |Dg(y)| dy \right] + 2(2^{3(n-1)}) \int_{A(2^{-k-1}, 2^{-k+2})} |Dg(y)| dy \\ &\leq 2^{3n} \int_{A(2^{-k-1}, 2^{-k+2})} |Dg(y)| dy. \end{aligned}$$

Therefore,

$$\begin{aligned}
\int_E |DM^I f(y)| dy &\leq \sum_{k \in \mathbb{Z}} \int_{E_k} |DM^I f(y)| dy \\
&\leq 2^{3n} \sum_{k \in \mathbb{Z}} \int_{A(2^{-k-1}, 2^{-k+2})} |Dg(y)| dy \\
&= 3(2^{3n}) \sum_{k \in \mathbb{Z}} \int_{A(2^{-k}, 2^{-k+1})} |Dg(y)| dy = 3(2^{3n}) \|Dg\|_1.
\end{aligned}$$

This completes the proof.  $\square$

Then we are ready to prove our main theorem.

**Theorem 3.11.** *If  $f \in W^{1,1}(\mathbb{R}^n)$  is radial, then  $Mf \in W^{1,1}(\mathbb{R}^n)$  and  $\|DMf\|_1 \leq C_n \|Df\|_1$ .*

*Proof.* Let

$$E := \{x \in \mathbb{R}^n : Mf(x) > M^I f(x), DMf(x) \neq 0\}.$$

It is well known that  $Mf$  is locally Lipschitz in  $\{Mf(x) > f(x)\}$ , implying the existence of  $DMf$  in  $\{Mf(x) > f(x)\}$ . Since  $Mf \geq M^I f(x)$ , it holds that  $Mf(x) = \max\{Mf(x), M^I f(x)\}$ . Therefore, the theorem follows by Lemmas 3.4 and 3.2, if we can show that

$$\int_E |DMf(y)| dy \leq C_n \|Df\|_1. \quad (3.18)$$

To show this, observe first that for all  $x \in E$  there exist  $r_x > \frac{|x|}{4}$  and  $z_x \in \mathbb{R}^n$  such that  $x \in B(z_x, r_x) \in \mathcal{B}_x$ . Moreover, since  $DMf(x) \neq 0$ , Lemma 2.2 ((2) and (3)) says that  $x \in \partial B(z_x, r_x)$  and  $DMf(x)/|DMf(x)| = (z_x - x)/|z_x - x|$ . On the other hand,  $Mf$  is radial and so  $DMf(x)/|DMf(x)| = \pm x/|x|$ . We conclude that

$$B_x = B(c_x x, |c_x x - x|) \text{ for some } c_x \in \mathbb{R}.$$

Observe that  $r_x = |c_x x - x| = |c_x - 1||x| > |x|/4$  by the assumption, and thus  $|c_x - 1| > 1/4$ . Moreover, it holds that  $c_x \geq -1$ . To see this, observe that if  $c_x < -1$ , then  $-x \in B_x$  and, since  $Mf$  is radial,  $B_x \in \mathcal{B}_{-x}$ , implying by Lemma 2.2 that  $0 = DMf(-x) = DMf(x)$ , which contradicts with the assumption  $x \in E$ . Summing up, we can write  $E = E_+ \cup E_-$ , where

$$E_+ = \{x \in E : c_x > 1 + 1/4\} \text{ and } E_- = \{x \in E : -1 \leq c_x < 3/4\}.$$

We are going to use different estimates for  $DMf(x)$  in  $E_+$  and  $E_-$ . Since  $|DMf(x)| = |DMf(x) \cdot \frac{x}{|x|}|$ , it follows from Lemma 2.2 (2.9) that

$$|DMf(x)| \leq \frac{1}{|x|} \int_{B_x} |Df(y)| |y| dy.$$

This estimate will be used in  $E_-$ , while in  $E_+$  we will use (easier) estimate  $|DMf(x)| \leq \int_{B_x} |Df|$  (Lemma 2.2, (1)). We get that

$$\begin{aligned} \int_E |DMf(x)| dx &\leq \int_E \chi_{E_+}(x) |DMf(x)| + \chi_{E_-}(x) |DMf(x)| dx \\ &\leq \int_E \chi_{E_+}(x) \left( \int_{B_x} |Df|(y) dy \right) + \chi_{E_-}(x) \left( \int_{B_x} |Df|(y) \frac{|y|}{|x|} dy \right) dx \\ &= \int_E \int_{\mathbb{R}^n} \frac{\chi_{E_+}(x) \chi_{B_x}(y) |Df|(y)}{|B_x|} + \frac{\chi_{E_-}(x) \chi_{B_x}(y) |Df|(y) |y|}{|B_x| |x|} dy dx \\ &= \int_{\mathbb{R}^n} |Df|(y) \left( \int_{E_+} \frac{\chi_{B_x}(y)}{|B_x|} dx + \int_{E_-} \frac{\chi_{B_x}(y) |y|}{|B_x| |x|} dx \right) dy. \end{aligned}$$

If  $y \in B_x$  and  $x \in E_+$ , it follows from the definition of  $E_+$  that  $|x| \leq |y|$ . Moreover,  $y \in B_x$  and  $x \in E$  imply also that  $r_x \geq \max\{|y - x|, \frac{|x|}{4}\} \geq \frac{|y|}{6}$ . This implies the estimate

$$\int_{E_+} \frac{\chi_{B_x}(y)}{|B_x|} dx \leq \int_{B(0, |y|)} \frac{dx}{\omega_n(|y|/6)^n} \leq C_n, \text{ for all } y \in \mathbb{R}^n.$$

On the other hand, if  $x \in E_-$ , then  $-1 \leq c_x < 3/4$  especially implies that  $B_x \subset B(0, |x|)$ . Therefore, if  $x \in E_-$  and  $y \in B_x$ , then  $y \in B(0, |x|)$ , and thus  $|x| \geq |y|$ . Recall also that  $r_x \geq \frac{|x|}{4}$ . Combining these yields that

$$\int_{E_-} \frac{\chi_{B_x}(y) |y|}{|B_x| |x|} dx \leq |y| \int_{\mathbb{R}^n \setminus B(0, |y|)} \frac{dx}{\omega_n(|x|/4)^{n+1}} = C'_n |y| \int_{|y|}^{\infty} \frac{dt}{t^2} = C'_n,$$

for all  $y \in \mathbb{R}^n$ . This completes the proof.  $\square$

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF JYVÄSKYLÄ,  
P.O.Box 35 (MAD), 40014 UNIVERSITY OF JYVÄSKYLÄ, FINLAND

*E-mail address:* `hannes.s.luiro@jyu.fi`